

The Holographic Bound in Anti-de Sitter Space

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The correspondence between string theory in Anti-de Sitter space and super Yang Mills theory is an example of the Holographic principle according to which a quantum theory with gravity must be describable by a boundary theory. However, arguments given so far are incomplete because, while the bulk theory has been related to a boundary theory, the holographic bound saying that the boundary theory has only one bit of information per Planck area has not been justified. We show here that this bound is the physical interpretation of one of the unusual aspects of the correspondence between Anti-de Sitter space and the boundary conformal field theory, which is that infrared effects in the bulk theory are reflected as ultraviolet effects in the boundary theory.

1. Introduction

According to the holographic [1,2] hypothesis, a macroscopic region of space and everything inside it can be represented by a boundary theory living on the boundary of the region. Furthermore, the boundary theory should not contain more than one degree of freedom per Planck area. More precisely, the number of distinct quantum states should not exceed $\exp \frac{A}{4G_D}$. Here A represents the $d - 1$ dimensional area in a $d + 1 = D$ dimensional spacetime and G is the gravitational constant in D dimensions. One might imagine that the boundary theory is cutoff or discrete so that the information density is bounded.

Some recent support for this view has come from the study of Type IIB string theory on the background $AdS_5 \times S^5$, with a characteristic radius R for both factors and N units of five-form flux on S^5 . In particular, this theory appears to be dual to 3+1 dimensional $U(N)$ super Yang Mills theory with 16 real supercharges [3]. The super Yang Mills theory lives on the boundary of the AdS space. It has been possible [4,5] to describe a precise recipe expressing correlation functions of the boundary theory in terms of calculations performed in the bulk.

Though this equivalence of a bulk theory with gravity to a boundary theory without gravity is an important part of the holographic hypothesis, another important aspect has not yet been addressed in the literature. This is the holographic bound on the information density of the boundary theory: it should have only a finite number of degrees of freedom per Planck area. In fact, the counting requires some care, because in the usual form of the correspondence the entropy and area are both infinite. The entropy of the boundary theory is infinite, because this theory is a conformal field theory which has degrees of freedom at arbitrarily small scale. On the other hand, the boundary of the AdS space has infinite area. There is no contradiction, just a question of whether there is a natural way to regulate and compare these two infinities.

We will see that the essence of the matter has to do with the following fact about the correspondence between AdS space and the conformal field theory on the boundary: infrared effects in AdS space correspond to ultraviolet effects in the boundary theory. This shows up in many aspects of the correspondence between these two types of theory, going back to brane scattering computations [6] from which the equivalence was first guessed. For example [5], relevant, marginal, and irrelevant perturbations of the boundary conformal field

theory (which are perturbations that vanish, remain constant, or diverge as one goes to the ultraviolet), are mapped to perturbations of the AdS space that vanish, remain constant, or diverge as one goes to spatial infinity, that is to the infrared. As a variant of this (see the discussion of gravity in section 2.4 of [5]), ultraviolet divergences of the boundary theory in coupling to a background gravitational field, and the resulting conformal anomaly, are derived from an infrared divergence in computing the total volume of the interior. We will call this relation the I.R. - U.V. connection. The contribution of the present paper is to show that the I.R. - U.V. connection is the key to the information bound that is an important part of the holographic hypothesis.

We begin by explaining why the I.R. - U.V. connection is natural given the geometry of AdS space. Then we explain why it is the key to the “information bound” in the holographic hypothesis, and conclude with some general remarks on holography in AdS space.

2. AdS Space

There are many ways to present AdS space. For our purposes we find it particularly convenient to represent it as a product of a unit four dimensional spatial ball with an infinite time axis^{*}. The metric has the form

$$dS^2 = R^2 \left[\frac{4dx^i dx^i}{(1-r^2)^2} - dt^2 \frac{1+r^2}{1-r^2} \right] \quad (2.1)$$

where $i = 1, \dots, 4$ and $r^2 = x^i x^i$. The AdS space is the ball $r < 1$. The boundary conformal theory lives on the sphere $r = 1$.

We will use the notation x for points in the bulk of the space and X for points on the boundary. Our conventions will be as follows. When discussing the bulk theory, distances will mean proper distance as defined by (2.1). On the other hand when discussing the surface theory distances on the unit sphere will be defined to be dimensionless and given by the metric (2.1) without the factor R^2 . Similarly concepts such as temperature in the boundary theory will be defined to be dimensionless. The dimensions can be restored with the appropriate factors of R .

* AdS space is sometimes assumed to have a periodic time. In this paper we work on the covering space for which $-\infty < t < +\infty$. Unlike some other coordinates the coordinates we use cover the entire AdS in a single valued manner.

The correspondence between the bulk supergravity in the ball and the surface super Yang Mills theory requires a relationship between R , the radius of the AdS , and N , the size of the gauge group [3]:

$$R = l_s(g_s N)^{1/4} \quad (2.2)$$

Here g_s, l_s are the string coupling constant and string length scale.

The duality between the two theories is expressed in terms of correlators on the boundary. In particular, supergravity field correlators $G(x_1)G(x_2)$ of various kinds should be equal to super Yang Mills correlators $Y(X_1)Y(X_2)$ when the points x are brought to the boundary points X . Let us consider the behavior of these correlators in some more detail. We will assume that the fields Y are dimensionless. This can always be arranged by introducing an arbitrary regulator mass scale μ . Because the super Yang Mills theory is a conformal field theory, the operator products $Y(X_1)Y(X_2)$ should have the form

$$Y(X_1)Y(X_2) = \mu^{-p}|X_1 - X_2|^{-p} + \dots \quad (2.3)$$

for some p when the coordinate distance $|X_1 - X_2|$ on the unit sphere tends to zero.

In what follows we will regulate the area of the boundary by replacing it with a sphere just inside the boundary at $r = 1 - \delta$, where δ is a number much smaller than 1. The resulting area of the sphere is

$$A \approx \frac{R^3}{\delta^3} \quad (2.4)$$

Now consider the geodesic distance between two points X_1, X_2 on the regulated sphere (that is, the length of that part of a geodesic connecting X_1 and X_2 that lies at $r < 1 - \delta$). One easily finds that it is of order $\log(|X_1 - X_2|/\delta) = \log |X_1 - X_2| - \log \delta$. (In fact, the δ dependence comes from the divergence of the length of the geodesic as $\delta \rightarrow 0$; the dependence on $|X_1 - X_2|$ then follows on dimensional grounds.) A typical propagator for a particle of mass m in the bulk theory therefore has the behavior

$$\Delta(X_1, X_2) = \exp m[\log \delta - \log |X_1 - X_2|] = \frac{\delta^m}{|X_1 - X_2|^m} \quad (2.5)$$

for $|X_1 - X_2| \gg \delta$. For distances of order δ or smaller the power law is not correct. The effective theory on the regulated sphere is modified at small distances.

Comparing (2.3) and (2.5), we first of all see how it is possible for massive propagators in the bulk theory to be represented by power laws in the conformal theory. We also see that the infrared regulator δ in the bulk theory also plays the role of an ultraviolet regulator in the boundary super Yang Mills theory. Thus we see that regulating the large boundary area is represented by a short distance regulator in the super Yang Mills theory. We refer to this as the I.R.-U.V. connection.

Another example of this I.R. - U.V. connection can be seen by considering a string of the bulk theory which is stretched across a diameter of the ball. Its energy is easily computed as an integral along the string. One finds that it is linearly divergent near the boundary. The meaning of this is as follows. A string ending in the boundary is represented as a point charge in the super Yang Mills theory. The linearly divergent energy is the infinite self-energy of a point charge in 3 + 1-dimensional gauge theory. However, if we regulate the sphere then the linearly divergent energy becomes proportional to δ^{-1} , which is exactly what we would expect from a U.V. cutoff in the super Yang Mills theory.

3. Information and Cutoffs

In the last section, we saw that regulating the boundary area is equivalent to U.V. regulating the super Yang Mills theory. We will now make some intuitively plausible assumptions about the information storage capacity of a cutoff field theory. Introducing a cutoff in field theory can be viewed as replacing the space that the field theory lives in by discrete cells of the cutoff size. In this case we replace the regulated sphere by cells of coordinate size δ (δ is dimensionless). We assume that each independent quantum field is replaced by a single degree of freedom in each cell. We also assume that each degree of freedom is capable of storing a single bit of information.

The assumption can also be stated in terms of a limitation of the allowable states of the cutoff quantum field theory. For example we can limit the energy density carried by a single quantum field to be no larger than δ^{-4} . Another possibility is to limit the local temperature to be less than δ^{-1} . All of these give the same answer, so we will simply say that there is one degree of freedom per cell per field degree of freedom. We are now ready to count.

The total number of cells making up the sphere is of order δ^{-3} , and the number of field degrees of freedom in a $U(N)$ theory is of order N^2 . Thus the number of degrees of freedom

is

$$N_{dof} = \frac{N^2}{\delta^3} \quad (3.1)$$

Using (2.4) we can write (3.1) as

$$N_{dof} = \frac{AN^2}{R^3} \quad (3.2)$$

Now using (2.2) we get

$$N_{dof} = \frac{AR^5}{l_s^8 g_s^2} \quad (3.3)$$

Finally we recognize $l_s^8 g_s^2 R^{-5}$ as the 5 dimensional gravitational constant so that we find

$$N_{dof} = A/G_5 \quad (3.4)$$

Apart from the numerical constant $1/4$, this is the desired result.

As a further illustration of the connection between the U.V. cutoff of the super Yang Mills theory and the large area regulator in the bulk supergravity theory, let us consider a thermal state of the super Yang Mills theory at temperature T . Such a thermal ensemble must represent an *AdS* Schwarzschild black hole centered at the center of the ball [5]. The super Yang Mills theory equation of state [7] has the form

$$S = N^2 T_{ym}^3 \quad (3.5)$$

where T_{ym} is the dimensionless temperature of the super Yang Mills theory. Using (2.2) and $T = T_{ym}/R$ we find

$$(TR)^3 = \frac{S g_s^2 l_s^8}{R^8} \quad (3.6)$$

which in fact is the correct connection between temperature and entropy for AdS black holes. If we now use the usual Bekenstein formula for the connection between area and black hole

entropy we find

$$T_{ym}^3 = A/R^3 \tag{3.7}$$

Now let us suppose that the super Yang Mills theory is regulated by saying that the maximum value of T_{ym} is $1/\delta$. This corresponds to a black hole of maximum area given by

$$A_{max} = R^3/\delta^3 \tag{3.8}$$

This is the area of a sphere $r = 1 - \delta$ which is indeed the largest sphere that is allowed in the regulated theory. Thus we again see how a U.V. cutoff in the boundary theory is connected to an I.R. cutoff in the bulk theory.

4. General Remarks On Holography

We will close with some comments about the nature of the mapping from the bulk to the boundary theory. In a conformally invariant theory, one can not only move things around in position; one can also rescale them. For example, if the theory has closed strings, then it must have strings of every size. The scale size of the string or other object is the degree of freedom which becomes the coordinate perpendicular to the boundary.[★] In particular, the I.R. - U.V. connection suggests that small (big) in the super Yang Mills theory sense means near the boundary (center) of the ball. This is easy to see directly in the *AdS* space. Take a region of size a near near the center of the ball. Now transport it by a conformal transformation (that is, by an element of the *AdS* symmetry group $SO(4, 2)$ or $SO(5, 1)$) to a point at a coordinate distance δ from the boundary. The region will be shrunk to a coordinate size δa . Thus it seems that scale size gets transmuted into a spatial dimension. The one thing which is far from obvious is why the super Yang Mills theory should behave locally in the scale size.

To summarize, we have shown that when suitably regulated, the super Yang Mills theory boundary theory provides a true holographic description including the bound of one bit per

★ This becomes clear if one represents *AdS* space by the metric $ds^2 = (1/(x^0)^2)((dx_0)^2 + \sum_{i=1}^4(dx^i)^2)$, where x^i are the boundary coordinates, and x^0 controls the distance from the boundary. A dilation of the boundary is generated by the isometry $x^i \rightarrow \lambda x^i$, $x^0 \rightarrow \lambda x^0$ of *AdS* space; so rescalings of an object change the distance from the boundary.

Planck area. To quantify just how strange and wonderful this is, consider the number of degrees per unit volume. If we had a holographic theory in an asymptotically flat space with spatial dimension d , we would simply reason that the number of degrees of freedom within a volume V (granted an ultraviolet cutoff at the Planck scale) would be proportional to V in an ordinary local field theory, but to $A = V^{(d-1)/d}$ in a holographic theory. Simply because $A/V \rightarrow 0$ for $V \rightarrow \infty$, holography entails a drastic reduction of the number of degrees of freedom. The same argument cannot be made quite as easily in a world of negative cosmological constant, since if one keeps fixed the radius R of curvature, then A and V are proportional to one another for $V \rightarrow \infty$. In fact, the relation between them is asymptotically $V = AR$ in AdS space, or $V = AR^6$ for $AdS_5 \times S^5$.

However, we can see the dramatic effects of holography if we vary the N of the boundary $U(N)$ gauge theory and thus let R vary. The relation $V = AR^6$ becomes using (3.3)

$$\frac{N_{dof}}{V} = \frac{A}{Rl_s^8 g_s^2} \quad (4.1)$$

As R becomes large the number of degrees of freedom per unit volume tends to zero. Nevertheless, the theory is capable of describing string theory with a length scale l_{st} in the bulk space that is independent of R .

Another way to make the same point is to consider the high temperature behavior of the entropy. A local field in AdS_n has an entropy of order T^{n-1} at high temperature, as noted in [8]. This is the standard high temperature scaling in $n - 1$ space dimensions; a salient point is that the coefficient of T^{n-1} is finite because of a red-shifting of the local temperature pointed out in [8]. In AdS_5 , this would give T^4 for the entropy of a local field at high temperatures. In $AdS_5 \times S^5$, a local field would have high temperature entropy of order T^9 . But the boundary conformal field theory has a high temperature entropy that is only of order T^3 for large T , showing that a holographic theory has much lower entropy than a local field theory in the same spacetime.

Given any theory in AdS spacetime, whether it contains gravity or not, one can define local operators in a putative boundary theory by considering, just as in [4,5], the boundary behavior of perturbations in the bulk theory. In this way, given any AdS theory, one can construct candidate correlation functions of a boundary theory. Is the bulk theory equivalent to the boundary theory that has these correlation functions? The answer is evidently “no,” if

the bulk theory does not have gravity, since the entropies disagree at high T , or perhaps more intuitively, since the bulk theory (being a local field theory without gravity) has ordinary local observables in the interior as well as local observables on the boundary. It is quite possible, however, that when the bulk theory has gravity, the answer is always “yes.” This hypothesis is a sharpened form of the holographic hypothesis, for the case of theories with a negative cosmological constant.

It remains to ask whether one can build a similarly sharpened holographic hypothesis for theories with zero (or even positive) cosmological constant. The answer will require some new ideas, since Minkowski space (or de Sitter space) has no obvious close analog of the “boundary at spatial infinity” by which holography is realized when the cosmological constant is negative.

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